Firstly, the possibility of estimating the error in determining the heat-flux density by means of the analytic expressions (15), (17), (19), which affords a possibility of constructing heat meters with given accuracy. Secondly, simple computational formulas permitting the realization of continuous measurement of nonstationary heat fluxes during experiment by using analog facilities.

NOTATION

$p_k = \int_0^L dx \int_0^x Q_{m-1}^{(k)}(x)dx$; $p_k^* = \int_L^0 dx \int_L^{L-x} Q_{m-1}^{(k)}(L-x)dx$, weighting factors; $y_1 = \Delta T(\tau)$; $y_2 = \frac{\partial}{\partial \tau} \int_0^L dx \int_0^x t(x, \tau) dx$; σ_{y_1}, error in measuring $\Delta T(\tau)$; σ_{y_2}, error in determining y_2; a, thermal diffusivity coefficient; σ_0, methodological error in determining y_2; σ_k, error in measuring $T_k(\tau)$; ε, error in determining the derivative due to piecewise-linear interpolation; and σ_d, error in approximate analog differentiation.

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DERIVING THE THERMAL CONTACT RESISTANCE FROM THE SOLUTION OF THE INVERSE HEAT-CONDUCTION PROBLEM

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The construction of an iterative computational algorithm is considered, and results of mathematical modeling of the solution of the coefficient inverse problem of heat conduction by deriving the dependence of the thermal contact resistance on the temperature are given.

Consider the process of heat conduction in a two-layer infinite plate with known thermophysical characteristics of the layers and specified initial and boundary conditions of the first kind.

In real situations, there is contact heat transfer between the layers at the boundary. This means that, in numerical modeling of the heat-conduction process in the system, the energy-matching relations at the boundary between the layers must be considered, taking account of contact thermal resistance [1]. It is assumed that the heat conduction in each layer is described by a homogeneous heat-conduction equation. Then the mathematical formulation of the problem of heat conduction in a two-layer plate takes the following form for the given case

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$$C_i(T)\frac{\partial T_i(x,\tau)}{\partial\tau}=\frac{\partial}{\partial x}\left(\lambda_i(T)\ \frac{\partial T_i(x,\tau)}{\partial x}\right), X_i<x<X_{i+1},\ 0<\tau<\tau_m, \tag{1}$$

$$i=1,\ 2,$$

$$T_i(x,\ 0)=\varphi_i(x),\ X_i<x<X_{i+1},\ i=1,\ 2, \tag{2}$$

$$T_1(X_1,\ \tau)=q_1(\tau), \tag{3}$$

$$\lambda_1(T)\frac{\partial T_1(X_2,\tau)}{\partial x}=\lambda_2(T)\frac{\partial T_2(X_2,\tau)}{\partial x}, \tag{4}$$

$$-\lambda_1(T)\frac{\partial T_1(X_2,\tau)}{\partial x}R(T)=T_1(X_2,\ \tau)-T_2(X_2,\ \tau), \tag{5}$$

$$T_2(X_3,\ \tau)=q_2(\tau), \tag{6}$$

where X_1 is the coordinate of the beginning of the two-layer plate ($X_1 = 0$); X_2 is the thickness of the first layer; X_3 is the plate thickness.

In practical thermophysical investigations, the case when the contact thermal resistance is specified with very low accuracy or is completely unknown often arises. Therefore, a sufficiently accurate determination of the thermal contact resistance is necessary, since at large heat-flux densities the temperature difference in the contact region may be tens or even hundreds of degrees [1].

This problem may be regarded as an inverse heat-conduction problem: it is required to identify (estimate or refine) the characteristic R(T) of the mathematical model in Eqs. (1)-(6) from the results of observations in a system in conditions of real functioning. Observation is understood to mean the measurement of the given system's output parameters in experimental conditions, i.e., the determination of the system's reaction to some thermal perturbation.

In practical applications, it is simpler to make temperature measurements; therefore the case when it is temperature measurements which are made will be considered. Thus, it is necessary to determine the function R(T) in the mathematical model in Eqs. (1)-(6) from the results of measuring the temperature at a series of internal points of a two-layer plate. Note that a series of factors influences the thermal contact resistance [1, 2]. In the present work, the dependence of this characteristic on the temperature alone is considered. The influence of other factors may be investigated by the well-known traditional methods of [2].

Questions of the existence and uniqueness of a linear inverse problem of this kind for the model of a two-layer plate were analyzed in [3]. The dependence R(T) is most expediently determined using the apparatus of coefficient inverse problems or identification problems. In this approach, the desired dependence may be obtained directly from the data of a single experiment over a broad temperature range.

The choice of the desired characteristics is made on the basis of minimizing a particular criterion: the mean square deviation of the temperatures calculated from the specific mathematical model at the temperature-sensor positions from the experimental values.

The following notation is now introduced: m_i is the number of temperature sensors in the i-th layer ($i = 1, 2$); $Y_{i,j}$ is the coordinate of the position of the j-th temperature sensor ($j = 1, 2, \ldots, m_i$) in the i-th layer. In this notation, the inverse problem is formulated as follows: determine the functions R(T) and $T_i(x, \tau)$, $i = 1, 2$, from the condition of a minimum of the functional

$$J=\sum_{i=1}^{2}\sum_{j=1}^{m_i}\int_0^{\tau_m}(T_i(Y_{i,j},\ \tau)-f_{i,j}(\tau))^2\,d\tau, \tag{7}$$

where $f_{i,j}(\tau)$ is the measured time dependence of the temperature at the point with coordinates $Y_{i,j}$, while the function $T_i(x, \tau)$ satisfies the boundary problem in Eqs. (1)-(6). Using the approach developed in [4], the iterative algorithm for the solution of the formulated optimization problem is constructed. To this end, formulas for the target-functional gradient are obtained.

It is assumed that the function R(T) takes a small increment ΔR. Then the temperature $T_i(x, \tau)$, i = 1, 2, takes a small increment $\vartheta_i(x, \tau)$, $i = 1, 2$. It may be shown that the functions $\vartheta_i(x, \tau)$ satisfy the following boundary problem

$$C_i(T)\frac{\partial\vartheta_i(x,\tau)}{\partial\tau} = \lambda_i(T)\frac{\partial^2\vartheta_i(x,\tau)}{\partial x^2} + 2\frac{\partial T_i(x,\tau)}{\partial x}\frac{\partial\lambda_i(T)}{\partial T}\frac{\partial\vartheta_i(x,\tau)}{\partial x} + \left(\frac{\partial^2 T_i(x,\tau)}{\partial x^2}\frac{\partial\lambda_i(T)}{\partial T} + \left(\frac{\partial T_i(x,\tau)}{\partial x}\right)^2\frac{\partial^2\lambda_i(T)}{\partial T^2} - \frac{\partial T_i(x,\tau)}{\partial x}\times \frac{\partial C_i(T)}{\partial T}\right)\vartheta_i(x,\tau),\; X_i < x < X_{i+1},\; i = 1,2,\; 0 < \tau < \tau_m, \tag{8}$$

$$\vartheta_i(x, 0) = 0,\; X_i < x < X_{i+1},\; i = 1,2, \tag{9}$$

$$\vartheta_1(0, \tau) = 0, \tag{10}$$

$$\lambda_1(T)\frac{\partial\vartheta_1(X_2,\tau)}{\partial x} + \frac{\partial\lambda_1(T)}{\partial x}\frac{\partial T_1(X_2,\tau)}{\partial x}\vartheta_1(X_2,\tau) = \lambda_2(T)\frac{\partial\vartheta_2(X_2,\tau)}{\partial x} + \frac{\partial\lambda_2(T)}{\partial T}\frac{\partial T_2(X_2,\tau)}{\partial x}\vartheta_2(X_2,\tau), \tag{11}$$

$$R(T)\lambda_1(T)\frac{\partial\vartheta_1}{\partial x}(X_2,\tau) + \left(\lambda_1(T)\frac{\partial T_1(X_2,\tau)}{\partial x}\frac{\partial R(T)}{\partial T} + R(T)\frac{\partial\lambda_1(T)}{\partial T}\times \frac{\partial T_1(X_2,\tau)}{\partial x}\right)\vartheta_1(X_2,\tau) + \lambda_1(T)\frac{\partial T_1(X_2,\tau)}{\partial x}\Delta R + \vartheta_1(X_2,\tau) - \vartheta_2(X_2,\tau) = 0; \tag{12}$$

$$\vartheta_2(X_3, \tau) = 0. \tag{13}$$

The linear part of the increment in the target functional in Eq. (7) takes the form

$$\Delta J = 2\sum_{i=1}^{2}\sum_{j=0}^{m_i}\int_0^{\tau_m}(T_i(Y_{i,j},\tau) - f_{i,j}(\tau))\vartheta_i(Y_{i,j},\tau)\,d\tau. \tag{14}$$

The boundary problem conjugate with Eqs. (1)-(6) is now introduced into consideration

$$-C_i(T)\frac{\partial\psi_{i,j}(x,\tau)}{\partial\tau} = \lambda_i(T)\frac{\partial^2\psi_{i,j}(x,\tau)}{\partial x^2},\quad 0\leqslant\tau<\tau_m,\quad Y_{i,j-1}<x<Y_{i,j},\quad j = 1,2,\ldots, m_{i+1},\; i = 1,2, \tag{15}$$

$$\psi_{i,j}(X, \tau_m) = 0,\; j = 1,2,\ldots, m_i + 1,\; i = 1,2, \tag{16}$$

$$\psi_{1,1}(X_1, \tau) = 0, \tag{17}$$

$$\psi_{i,j}(Y_{i,j},\tau) = \psi_{i,j+1}(Y_{i,j},\tau),\; j = 1,2,\ldots, m_i,\; i = 1,2, \tag{18}$$

$$\frac{\partial\psi_{i,j}(Y_{i,j},\tau)}{\partial x} = \frac{\partial\psi_{i,j+1}(Y_{i,j},\tau)}{\partial x},\; j = 1,2,\ldots, m_i,\; i = 1,2, \tag{19}$$

$$-\lambda_2(T)R(T)\frac{\partial\psi_{2,1}(X_2,\tau)}{\partial x} = \psi_{1,m_1+1}(X_2,\tau) - \psi_{2,1}(X_2,\tau), \tag{20}$$

$$-\lambda_1(T)R(T)\frac{\partial\psi_{1,m_1+1}(X_2,\tau)}{\partial x} = \left(1 + \lambda_1(T)\frac{\partial T_1(X_2,\tau)}{\partial x}\frac{\partial R(T)}{\partial T}\right)(\psi_{1,m_1+1}(X_2,\tau) - \psi_{2,1}(X_2,\tau)), \tag{21}$$

$$\psi_{2,m_2+1}(X_3, \tau) = 0. \tag{22}$$

Using Eqs. (8)-(13) and (15)-(22), the expressions for the linear part of the increment in the functional in Eq.(16) may be transformed to give

$$\Delta J = \int_0^{\tau_m}\Delta R\lambda_1(T)\lambda_2(T)\frac{\partial\psi_{2,1}(X_2,\tau)}{\partial x}\frac{\partial T_1(X_2,\tau)}{\partial x}\,d\tau. \tag{23}$$

The region of definition of the function R(T) is not known a priori, and therefore it will be sought in the interval $D = (T_{min}, T_{max})$, which is realized in the plate and known in view of the homogeneity of Eq. (1) and the specified initial conditions and boundary conditions of the first kind [5]. The interval D is divided into l equal sections and a grid is introduced

$$\omega = \{T_k = T_{\min} + k\Delta T,\; k = -2, -1, \ldots, l + 3;\; \Delta T = (T_{\max} - T_{\min})/l\}.$$

The unknown function is approximated in the form of a B spline over the grid [6]

$$R(T)=\sum_{k=-1}^{l+1} r_k B_k(T),$$

where $B_k(T)$ is taken, so as to be specific, as a cubic B_0 spline [6]. Then, using Eq. (23), the formula for the components of the gradient vector in Eq. (7) may be written in the form

$$J_k'=\int_0^{\tau_m} \lambda_1(T)\lambda_2(T)\frac{\partial\psi_{2,1}(X_2,\tau)}{\partial x}\frac{\partial T_1(X_2,\tau)}{\partial x}B_k(T)\,d\tau,$$

$$k=-1,0,\ldots,\ l+1. \tag{24}$$

Knowing the gradient of the target functional, an iterative algorithm for the solution of the inverse problem may be constructed using the conjugate-gradient method [7]

$$r_k^{p+1}=r_k^p+\alpha_p g_k^p,\ k=-1,0,\ldots,l+1,\ p=0,1,\ldots, \tag{25}$$

where

$$g_k^p=-J_k'^{(p)}+\beta_p g_k^{p-1};\ \beta_0=0;\ \beta_p=\sum_{k=-1}^{l+1}\left(J_k'^{(p)}-J_k'^{(p-1)}\right)J_k'^{(p)}\Big/\sum_{k=-1}^{l+1}\left(J_k'^{(p)}\right)^2,\ p=1,2,\ldots.$$

The coefficient α_p determines the value of the step in the p-th iteration and is calculated from the condition min $J(\bar{r}^p+\alpha_p\bar{G}^p)$, where $\bar{r}_p$ is a vector of dimensionality $l+3$ corresponding to the values of the coefficients in the approximation in Eq.(24) in the p-th iteration and $\bar{G}^p$ is a vector of dimensionality $l+3$, the components of which are g_k in Eq. (25).

The value of the linear estimate α_p may be obtained in explicit form (see [4, 8], for example)

$$\alpha_p=-\sum_{i=1}^{2}\sum_{j=1}^{m_i}\frac{\int_0^{\tau_m}(T_i(Y_{i,j},\tau)-f_{i,j}(\tau))\vartheta_i(Y_{i,j},\tau)\,d\tau}{\int_0^{\tau_m}(\vartheta_i(Y_{i,j},\tau))^2\,d\tau}. \tag{26}$$

The iterative process is constructed as follows. The initial approximation of the desired parameters is constructed, and the problem in Eqs. (1)-(6) is solved. In the appropriate approximation of the temperature field, after solving the conjugate problem in Eqs. (15)-(22), the gradient of the target functional is calculated from Eq. (24). Then, after solving the problem for the increment in Eqs. (8)-(13), an estimate of the depth of descent in α_p is made, and a new approximation is found from Eq. (25). The process is then repeated. It is expedient to arrange for exit from the iterative process to occur on the basis of the discrepancy, i.e., when the condition $J\leqslant\delta^2$ is satisfied, where $\delta^2=\sum_{i=1}^{2}\sum_{j=1}^{m_i}\int_0^{\tau_m}\sigma_{i,j}^2(\tau)\,d\tau$ is the integral error in specifying the temperature at the temperature-sensor sites; $\sigma_{i,j}(\tau)$ is the mean square deviation of the input temperatures.

This algorithm was realized in the form of a program for the EC computer, and used in calculating a series of methodological examples.

Consider a plate consisting of two identical layers with a temperature sensor in each layer. The boundary temperatures and input data for the solution of the inverse problem are taken in the form of the temperatures obtained from the solution of the direct heat-conduction problem, with boundary conditions of the second kind and a specified thermal resistance $R(T) = T^2$. The other initial data are taken in the form

$$q_1(\tau)=1,\ q_2(\tau)=0,\ C_1(T)=C_2(T)=1,\ \lambda_1(T)=\lambda_2(T)=1,$$

$$X_2=0.5,\ X_3=1;\ Y_{1,1}=0.25,\ Y_{2,1}=0.75,\ T=(x,\ 0)=0.$$

The desired function R(T) is approximated using seven divisions of the temperature interval (T_{min}, T_{max}). The calculations are performed on a difference grid $n_x \times n_\tau = 20 \times 20$. In the given methodological example, the inverse problem was solved for "accurate" data obtained from the solution of the boundary problem in Eqs. (1)-(6). The integral error of the

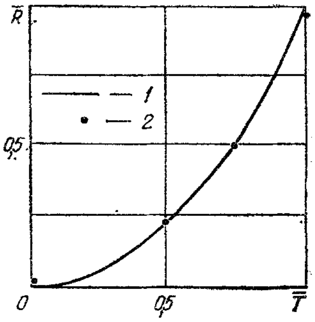

Fig. 1. Deriving the thermal contact resistance from the solution of the inverse heat-conduction problem: 1) specified dependence; 2) derived values ($\bar{T} = T/T_{max}$; $\bar{R} = R/R_{max}$).

temperature measurements was taken to be zero, and iteration was ended when the solution obtained in two successive intervals was the same. The initial approximation of the thermal resistance was taken to be constant: $R_0 = 0.1$.

The results of mathematical modeling show (Fig. 1) the possibility of using the proposed algorithm for analyzing and interpreting real experimental data.

NOTATION

T, temperature; C(T), bulk specific heat; $\lambda(T)$, thermal conductivity; x, coordinate; τ, time; τ_m, length of process; R(T) contact thermal resistance; $f_{i,j}(\tau)$, input temperatures; $\vartheta(x, \tau)$, temperature increment; r_k, $k = -1, 0, \ldots, m + 1$, parameters in the spline approximation of the function R(T); B(T), B spline; α, β, parameters of the conjugate-gradient method; J', gradient of the target functional; $\psi(x, \tau)$, conjugate variable; δ^2, integral error of input data; p, number of iterations; $q(\tau)$, specific heat flux; $\varphi(\tau)$, temperature distribution function at initial instant. Indices: max, min, maximum and minimum values, respectively.

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